HUPD-9831

$O(a\alpha_s)$ matching coefficients for the ΔB=2 operators in the lattice static theory

K-I. Ishikawa, T. Onogi and N. Yamada
Department of Physics, Hiroshima University, Higashi-Hiroshima 739-8526, Japan

Abstract

We present the perturbative matching coefficient to $O(a\alpha_s)$ which relates the ΔB=2 operator in the continuum to that of the lattice static theory, which is important in the accurate extraction of the continuum value of the B_B from lattice simulations. The coefficients are obtained by the one-loop calculations in both of the continuum and lattice theory. We find that two new dimension seven operators appear at the $O(a\alpha_s)$ with the $O(1)$ coefficients. We also discuss the possible cancellation of $O(a\alpha_s)$ correction in the ratio $B_B = \langle\overline{B}|\mathcal{O}_L|B\rangle/((8/3)(f_B M_B)^2)$ qualitatively.

PACS number(s): 12.38.Gc, 12.39.Hg, 13.20.He, 14.40.Nd

I. INTRODUCTION

One of the most important issues in particle physics is the origin of masses and CP violation. CKM matrix elements are believed to play a key role to probe the physics behind it. Despite a lot of efforts in various approaches, the matrix element V_{td} which can be determined from $B^0 - \overline{B^0}$ mixing is still only poorly known because of a lack of accuracy in the involved hadronic matrix element. The hadronic matrix element is parameterized using the B meson decay constant f_B and the bag parameter B_B as $B_B f_B^2$, so it is quite crucial to compute them with high precision. For this purpose, the lattice QCD has been considered to be one of the most reliable approaches. So far most of the efforts have been devoted to the B meson decay constant. At the early stage, the decay constants in the static approximation and from the extrapolation from light quarks were computed. It was found that both the lattice cutoff dependence and heavy quark mass dependence are significantly large. Later the scaling behavior for the lattice spacing a [1,2] and the heavy quark mass $1/m_Q$ [3] have been investigated carefully and the best estimate on f_B from the quenched Lattice QCD is now $f_B = 165(20)$ MeV [4]. On the other hand, until recently, the bag parameter has been calculated only either in the static limit or by the naive extrapolation from light quarks. In this respect, careful studies of systematic errors of the bag parameter are still missing.

In general, in order to get a continuum result of the physical quantity such as f_B from lattice simulation, we have to compute physical quantities on different lattices and extrapolate the results to the continuum. Therefore the final results have smaller errors if the cutoff dependence is smaller. It was found that the $O(a)$ improvements of the action and lattice operators in Symanzik approach significantly reduce the lattice cutoff dependences of various matrix elements. For the heavy-light axial vector current, such kind of improvements have been accomplished by Morningstar and Shigemitsu [5] in the lattice NRQCD formalism. They found that the additional operator mixed at the $O(a\alpha_s)$ and the inclusion of the effect significantly reduced the value of f_B at the finite lattice spacing and it was also the case in the static limit. In contrast to the decay constant, the $O(a\alpha_s)$ mixing effect has not been studied for B_B. One reason is that only the operator matching of $O(\alpha_s)$ has been done in Refs. [6–9] so far. Although previous simulations have not shown a clear cutoff dependence of the B_B [9,10], it would be very important to study the $O(a\alpha_s)$ mixing effect explicitly in order to obtain the precise value of B_B.

The purpose of this paper is to investigate the $O(a\alpha_s)$ effect for the B_B. We perturbatively compute the operator matching coefficients of the static-clover ΔB=2 operators up to $O(a\alpha_s)$. We use the notation defined by the authors in Refs. [6,7].

Phenomenologically important quantity might be the product of $B_B f_B^2$ which is just the expectation value of ΔB=2 operator. Therefore it seems sufficient to improve only the ΔB=2 operator. To determine f_B and B_B separately, however, would have some more advantage from a technical point of view [4]. Since the $O(a\alpha_s)$ improvement for B_B requires the improvements of both of the heavy-light axial vector current and the ΔB=2 operator, we also mention the result for the heavy-light current for completeness.

The paper is organized as follows. In sections II and III, our main results, the matching coefficients to the $O(a\alpha_s)$ for the heavy-light current and the ΔB=2 operator, are shown, respectively. In section IV, we discuss the impact of our results on the determination of the B_B using the typical values of the parameters involved and the consistency with the

previous observations for the cutoff dependence of the B_B. Finally we conclude in section V. The appendices are devoted to some details in this calculation.

Throughout this paper, we choose Feynman gauge (α=1) and the light quark mass m_q is set to be zero. The ultraviolet divergences appearing in the continuum calculation are regulated by dimensional regularization and the continuum operators are renormalized with $\overline{\rm MS}$ scheme, while the infrared divergences are regulated by the gluon mass λ in both of the continuum and lattice theory. The operators with superscript "con" and "lat" define the continuum operators and the lattice operators, respectively. In our convention, γ_5 always anticommutes with γ_μ. We give all the equations in Euclidean form.

II. STATIC HEAVY-LIGHT CURRENT

In this section, we present the matching coefficients of the static-light current operators which are relevant to the determinations of the form factors of the static to light decays as well as the following discussion. Our lattice gauge action is the standard Wilson plaquette action. For the light quark we use the $O(a)$-improved SW quark action [11] with the clover coefficient $c_{\rm sw}$ and, in contrast to Ref. [7], we do not incorporate the rotation operator associated with the clover fermion in the current operator.

In the following, we describe the lattice static quark. In the static limit, the quark action is separated into two pieces in Dirac basis, namely one for the static quark b' and the other for the static antiquark $\tilde{b}'$. Both are two-component fields which are related to the relativistic four-component field b as

$$b = \begin{pmatrix} b' \\ \tilde{b}'^{\dagger} \end{pmatrix}, \quad \bar{b} = \begin{pmatrix} b'^{\dagger} & -\tilde{b}' \end{pmatrix}. \tag{1}$$

In our convention, the action is given by

$$\begin{aligned} S^{\rm stat} = & \sum_{x,y} b'^{\dagger i}_{\alpha}(x)[\ \delta_{x,y}\delta^{ij} - U_4^{\dagger ij}(y)\delta_{x-\hat{4},y}]\delta_{\alpha\beta} b'^{j}_{\beta}(y) \\ & + \sum_{x,y} \left(-\tilde{b}'^{i}_{\alpha'}(x)\right) [\ \delta_{x,y}\delta^{ij} - U_4^{ij}(x)\delta_{x+\hat{4},y}]\delta_{\alpha'\beta'} \tilde{b}'^{\dagger j}_{\beta'}(y), \end{aligned} \tag{2}$$

where α (α') and β (β') run over 1 and 2 (3 and 4). Our Feynman rules for the lattice static quark and antiquark are obtained from the above action through the standard procedure. The heavy quark (antiquark) propagates only forward (backward) in time direction on the lattice.

To determine the matching coefficients upto $O(a\alpha_s)$, (i) we calculate the heavy to light on-shell scattering amplitudes through the following operator with arbitrary gamma matrix Γ,

$$J_\Gamma^{(0)} = \overline{q}\Gamma b,$$

in the continuum full theory upto the one-loop order, expand the resulting expression with respect to the momenta of external quarks at their rest frame, which is required to obtain the matching coefficients through desired order $O(a\alpha_s)$, and take the static limit of the

heavy quarks. (ii) We repeat the similar calculation to step (i) on the lattice static theory. (iii)Finally we express the continuum operators in terms of the lattice operators with appropriate matching coefficients which are adjusted to coincide the one-loop scattering amplitude of both theories through $O(a\alpha_s)$. In this matching procedure, we have two coupling constants, $\alpha_s^{\overline{\rm MS}}$ in the continuum theory and $\alpha_s^{\rm lat}$ on the lattice theory. Through this paper, both coupling constants are rewritten in terms of the V-scheme coupling [13] at one-loop order.

According to step (i), we calculate the scattering amplitude with an initial heavy quark carrying momentum $\vec{p}$ and a final light quark carrying momentum $\vec{k}$. The resulting expression is

$$\begin{aligned}\langle q(\vec{k})|\ J_\Gamma^{(0)\rm con}\ |b(\vec{p})\rangle = \Bigg[\ 1+\frac{\alpha_s}{4\pi}C_F\ \Bigg(\left(\frac{1}{4}H^2-\frac{5}{2}\right)\ln\left(\frac{\mu^2}{m_b^2}\right) \\ -\frac{3}{2}\ln\left(\frac{\lambda^2}{\mu^2}\right)-\frac{HG}{2}+\frac{3}{4}H^2-HH'-\frac{11}{4}\Bigg)\Bigg]\langle J_\Gamma^{(0)}\rangle_0 \\ +\frac{\alpha_s}{4\pi}C_F\ G\ \frac{8\pi}{3a\lambda}\langle J_\Gamma^{(1)}\rangle_0,\end{aligned} \tag{3}$$

where the symbol $\langle\cdots\rangle_0$ denotes the tree level expectation value between the same initial and final states as those of the left hand side, $C_F=(N_c^2-1)/2N_c$ with number of color N_c, m_b is the heavy quark mass, and $J_\Gamma^{(1)}\equiv\overline{q}(a\overleftarrow{D}\cdot\vec{\gamma})\Gamma b$. The renormalization scale for the amplitude is μ. The definitions of H, G and H' are the same as those in Ref. [14]. In deriving Eq. (3), we use the equation of motion for the light quark, $\overline{q}\gamma_4 k_4=-\overline{q}\vec{\gamma}\cdot\vec{k}$ and also that for the heavy quark, $\gamma_4 u_b=u_b$, to simplify the result.

Repeating the similar calculation to the continuum theory according to step (ii), we obtain the corresponding amplitude on the lattice as follows.

$$\begin{aligned}\langle q(\vec{k})|\ J_\Gamma^{(0)\rm lat}\ |b(\vec{p})\rangle = \Bigg[1+\frac{\alpha_s}{4\pi}C_F\ \left(-\frac{3}{2}\ln(a^2\lambda^2)+A_\Gamma^{(0)}+A_\Gamma^{I(0)}+\frac{1}{2}u_0^{(2)}\right)\Bigg]\langle J_\Gamma^{(0)}\rangle_0 \\ +\frac{\alpha_s}{4\pi}C_F\ \left(G\ \frac{8\pi}{3a\lambda}+r\ (1-c_{\rm sw})\ln(a^2\lambda^2)+A_\Gamma^{(1)}+A_\Gamma^{I(1)}\right)\langle J_\Gamma^{(1)}\rangle_0,\end{aligned} \tag{4}$$

where

$$A_\Gamma^{(0)} = d_1+d_2G+\frac{1}{2}(e^{(R)}+f), \tag{5}$$

$$A_\Gamma^{I(0)} = -d^IG+\frac{1}{2}f^I, \tag{6}$$

$$A_\Gamma^{(1)} = UG+V, \tag{7}$$

$$A_\Gamma^{I(1)} = U^IG+V^I. \tag{8}$$

The renormalization scale for the amplitude is a^{-1}. $A_\Gamma^{(0)}$ and $A_\Gamma^{I(0)}$ correspond to A_Γ and A_Γ^I in Ref. [7], respectively, and the numerical values of d_1, d_2, $e^{(R)}$ and f are tabulated in Refs. [6,7,14]. Although our explicit form of the integrand of d^I completely agrees with that of Ref. [7], the numerical value of d^I is slightly larger in magnitude than that of Ref. [7], and the value is tabulated in Table I. U, U^I, V and V^I are new contributions at $O(a\alpha_s)$.

Their explicit forms of the integrands are shown in Appendix B and their numerical values are tabulated in Table I. The coefficients with the superscript I vanish when Wilson light quark is used ($c_{\rm sw}$=0), which is the same notation as Ref. [7]. $u_0^{(2)}$ comes from the tadpole improvement of the light quark wave function renormalization, for details see Appendix A.

In step (iii), matching Eq. (3) to Eq. (4), we obtain the following relation between the operators in the continuum and lattice theory,

$$\begin{aligned} J_\Gamma^{(0)\rm con} &= \left[1+\frac{\alpha_s}{4\pi}C_F\ \zeta_\Gamma^{(0)}\ \right] J_\Gamma^{(0)\rm lat} + \frac{\alpha_s}{4\pi}C_F\ \zeta_\Gamma^{(1)}\ J_\Gamma^{(1)\rm lat} \\ &\equiv Z_\Gamma^{(0)}\ J_\Gamma^{(0)\rm lat} + Z_\Gamma^{(1)}\ J_\Gamma^{(1)\rm lat}, \end{aligned} \tag{9}$$

where

$$\begin{aligned} \zeta_\Gamma^{(0)} &= \left(\frac{1}{4}H^2-\frac{5}{2}\right)\ln\left(\frac{\mu^2}{m_b^2}\right) - \frac{3}{2}\ln\left(\frac{\lambda^2}{\mu^2}\right) - \frac{HG}{2} + \frac{3}{4}H^2 - HH' - \frac{11}{4} \\ &\quad + \frac{3}{2}\ln(a^2\lambda^2) - A_\Gamma^{(0)} - A_\Gamma^{I(0)} - \frac{1}{2}u_0^{(2)} \end{aligned} \tag{10}$$

$$\zeta_\Gamma^{(1)} = -r\ (1-c_{\rm sw})\ln(a^2\lambda^2) - A_\Gamma^{(1)} - A_\Gamma^{I(1)} \tag{11}$$

Eq. (10) has been calculated in Refs. [6,7] except for the differences of our inclusion of tadpole improvements and the wave function renormalization of lattice static quarks. Eq. (11) is new result for the arbitrary static-light current. For axial vector current and vector current the matching coefficient for $J_\Gamma^{(1)\rm lat}$ has been calculated with NRQCD action for heavy quarks in Ref. [5]. From Eq. (9) we observe that the $O(a)$ operator $J_\Gamma^{(1)\rm lat}$ appears at this order, which is considered to be a lattice artifact. It is noted that there is no linear divergence proportional to $1/\lambda$ in the coefficients, while there is a logarithmic divergence unless $c_{\rm sw}=1$. In the use of Wilson light quark ($c_{\rm sw}$=0), therefore, we cannot match these operators consistently due to this infrared mismatch as previously pointed out in Refs. [5,7].

The results of $\zeta_\Gamma^{(0)}$ and $\zeta_\Gamma^{(1)}$ for each Γ are summarized in Table II, where $r=c_{\rm sw}=1$ and the tadpole improvement is performed by using the perturbative expression of the critical hopping parameter. The numerical values of $O(a\alpha_s)$ correction for axial vector current and vector current are consistent with those in Ref. [5][1]. It should be noted that the coefficient of $J_\Gamma^{(1)\rm lat}$ depends only on G, $G=-1$ might lead to a large mixing effect, while $G=1$ does not. Actually the mixing effect leads to the significant change for the f_B, which has been seen in Refs. [2,15].

III. ΔB=2 OPERATOR

In this section, we discuss the matching of the ΔB=2 operator. The matching procedure is completely common as that for heavy-light current previously shown, and we follow the previous step. Before proceeding step (i), we give the definitions of the operators.

[1] Note that since there are some differences of the definitions of the lattice operators and the matching coefficients between ours and theirs in Ref. [5], one would need to redefine our definitions to compare the results with theirs.

$$
\begin{aligned}
\mathcal{O}_L &= \left[\overline{b}\gamma_\mu P_L q\right]\left[\overline{b}\gamma_\mu P_L q\right],\\
\mathcal{O}_S &= \left[\overline{b} P_L q\right]\left[\overline{b} P_L q\right],\\
\mathcal{O}_R &= \left[\overline{b}\gamma_\mu P_R q\right]\left[\overline{b}\gamma_\mu P_R q\right],\\
\mathcal{O}_N &= 2\left[\overline{b}\gamma_\mu P_L q\right]\left[\overline{b}\gamma_\mu P_R q\right] + 4\left[\overline{b} P_L q\right]\left[\overline{b} P_R q\right],\\
\mathcal{O}_{LD} &= \left[\overline{b}\gamma_\mu P_L q\right]\left[\overline{b}\gamma_\mu P_L (a\overrightarrow{D}\cdot\overrightarrow{\gamma}) q\right],\\
\mathcal{O}_{ND} &= 2\left[\overline{b}\gamma_\mu P_L q\right]\left[\overline{b}\gamma_\mu P_R (a\overrightarrow{D}\cdot\overrightarrow{\gamma}) q\right] + 4\left[\overline{b} P_L q\right]\left[\overline{b} P_R (a\overrightarrow{D}\cdot\overrightarrow{\gamma}) q\right],
\end{aligned}
$$

where $P_L{=}1-\gamma_5$ and $P_R{=}1+\gamma_5$.

According to step (i), we calculate the two-body scattering amplitude for $\mathcal{O}_L$ between the initial state with a heavy antiquark and a light quark and the final state with a heavy quark and a light antiquark in the continuum theory. The initial heavy antiquark carries momentum $\vec{p}_2$, the initial light quark $\vec{k}_2$, the final heavy quark $\vec{p}_1$, and the final light antiquark $\vec{k}_1$. We obtain the scattering amplitude in the continuum theory at one-loop as

$$
\begin{aligned}
&\langle \overline{q}(\vec{k}_1), b(\vec{p}_1)|\ \mathcal{O}_L^{\rm con}\ |q(\vec{k}_2), \overline{b}(\vec{p}_2)\rangle = Z_q^{\rm con} Z_b^{\rm con} \sum_i \mathcal{V}^{(i)}_{\rm con}(\vec{k}_1, \vec{p}_1, \vec{k}_2, \vec{p}_2)\\
&= \left[1 + \frac{\alpha_s}{4\pi}\ \left(2\ln\left(\frac{m_b^2}{\mu^2}\right) - 4\ln\left(\frac{\lambda^2}{m_b^2}\right) + C_L + \frac{7}{3}\right)\right]\langle\mathcal{O}_L\rangle_0\\
&\quad + \frac{\alpha_s}{4\pi}\ C_S\langle\mathcal{O}_S\rangle_0 + \frac{\alpha_s}{4\pi}\ \frac{16\pi}{3a\lambda}\langle\mathcal{O}_{ND}\rangle_0,
\end{aligned}
\tag{12}
$$

where the $\mathcal{V}^{(i)}_{\rm con}$ (i runs over a-d) denotes the contribution from each diagram in the continuum theory, which appear in Appendix C. The constants $C_L = -14$ and $C_S = -8$ appear in Refs. [6,7].

According to step (ii), we calculate the corresponding amplitude with the lattice theory and obtain the result as follows.

$$
\begin{aligned}
&\langle \overline{q}(\vec{k}_1), b(\vec{p}_1)|\ \mathcal{O}_L^{\rm lat}\ |q(\vec{k}_2), \overline{b}(\vec{p}_2)\rangle = Z_q^{\rm lat} Z_b^{\rm lat} \sum_i \mathcal{V}^{(i)}_{\rm lat}(\vec{k}_1, \vec{p}_1, \vec{k}_2, \vec{p}_2)\\
&= \left[1 + \frac{\alpha_s}{4\pi}\ \left(-4\ln(a^2\lambda^2) - D_L - D_L^I + \frac{7}{3} + \frac{4}{3}u_0^{(2)}\right)\right]\langle\mathcal{O}_L\rangle_0\\
&\quad + \frac{\alpha_s}{4\pi}\ (-D_N - D_N^I)\langle\mathcal{O}_N\rangle_0 + \frac{\alpha_s}{4\pi}\ (-D_R - D_R^I)\langle\mathcal{O}_R\rangle_0\\
&\quad + \frac{\alpha_s}{4\pi}\ \left(-\frac{10}{3}\ r(1-c_{\rm sw})\ln(a^2\lambda^2) - D_{LD} - D_{LD}^I\right)\langle\mathcal{O}_{LD}\rangle_0\\
&\quad + \frac{\alpha_s}{4\pi}\ \left(\frac{16\pi}{3a\lambda} - D_{ND} - D_{ND}^I\right)\langle\mathcal{O}_{ND}\rangle_0,
\end{aligned}
\tag{13}
$$

where

$$
D_L = -\frac{10}{3}d_1 - \frac{1}{3}c - \frac{1}{3}v - \frac{4}{3}(e^{(R)} + f) + \frac{7}{3}, \tag{14}
$$

$$
D_L^I = -\frac{1}{3}v^I - \frac{4}{3}f^I, \tag{15}
$$

$$
D_N = 2\ d_2, \tag{16}
$$

$$D_N^I = -2\ d^I, \tag{17}$$
$$D_R = \frac{4}{3} w, \tag{18}$$
$$D_R^I = \frac{4}{3} w^I, \tag{19}$$
$$D_{LD} = \frac{10}{3} V, \tag{20}$$
$$D_{LD}^I = \frac{10}{3} V^I, \tag{21}$$
$$D_{ND} = -2U, \tag{22}$$
$$D_{ND}^I = -2U^I. \tag{23}$$

The coefficients D_L, D_L^I, D_N, D_N^I, D_R, and D_R^I have been calculated in Refs. [6–9] and we use the same notation as those in Refs. [6,7] for convenience. The coefficients D_{LD}, D_{LD}^I, D_{ND}, and D_{ND}^I are novel results of this paper. $\mathcal{V}_{\rm lat}^{(i)}$ (i runs over a-d) is the contribution from each diagram in the lattice theory, which are shown in the Appendix C.

According to step (iii), using Eqs. (12) and (13) we match the the lattice operator and continuum one to $O(a\alpha_s)$. The obtained operator identity is

$$\mathcal{O}_L^{\rm con} = \sum_X Z_X\ \mathcal{O}_X^{\rm lat}, \tag{24}$$

where X runs over $\{L, S, N, R, LD, ND\}$,

$$Z_L = 1 + \frac{\alpha_s}{4\pi}\left(6\ln(a^2 m_b^2) - 2\ln(a^2\mu^2) + C_L + D_L + D_L^I - \frac{4}{3}u_0^{(2)}\right), \tag{25}$$
$$Z_S = \frac{\alpha_s}{4\pi}\ C_S, \tag{26}$$
$$Z_N = \frac{\alpha_s}{4\pi}\ \left(D_N + D_N^I\right), \tag{27}$$
$$Z_R = \frac{\alpha_s}{4\pi}\ \left(D_R + D_R^I\right), \tag{28}$$
$$Z_{LD} = \frac{\alpha_s}{4\pi}\ \left(\frac{10}{3} r\ (1 - c_{\rm sw}) \ln(a^2\lambda^2) + D_{LD} + D_{LD}^I\right), \tag{29}$$
$$Z_{ND} = \frac{\alpha_s}{4\pi}\ \left(D_{ND} + D_{ND}^I\right). \tag{30}$$

Here we omit the explicit arguments of μ and a^{-1}, which are introduced by the renormalization procedure, for the operators O_X and the coefficients Z_X without ambiguity. We find that the above results to $O(\alpha_s)$ agrees with those of Refs. [6,7] except for the coefficient D_R^I in Ref. [7] (see Appendix C). The correct value of D_R^I including double rotation operator has been already obtained in Refs. [8,9] and our D_R^I is consistent with them. Two new operators $\mathcal{O}_{LD}^{\rm lat}$ and $\mathcal{O}_{ND}^{\rm lat}$ mix at the $O(a\alpha_s)$. It should be noted that the coefficients of the new operators have completely common integrands to those of $J_\Gamma^{(1)\rm lat}$ in the heavy-light current. The use of Wilson light quark ($c_{\rm sw}$=0) leads to the mismatch of infrared behavior between continuum and lattice theory as in the case of heavy-light current.

When $c_{\rm sw} = r = 1$ and the tadpole improvement by the critical hopping parameter are chosen for the numerical estimate of Eqs. (25)-(30), Eq. (24) becomes

$$
\begin{aligned}
\mathcal{O}_L^{\rm con} = &\left[\ 1+\frac{\alpha_s}{4\pi}\left(6\ln(a^2m_b^2)-2\ln(a^2\mu^2)-35.15\right)\right]\mathcal{O}_L^{\rm lat} \\
&+\frac{\alpha_s}{4\pi}\ (-8)\ \mathcal{O}_S^{\rm lat}+\frac{\alpha_s}{4\pi}\ (-6.16)\,\mathcal{O}_N^{\rm lat}+\frac{\alpha_s}{4\pi}\ (-0.52)\,\mathcal{O}_R^{\rm lat} \\
&+\frac{\alpha_s}{4\pi}\ (-17.20)\,\mathcal{O}_{LD}^{\rm lat}+\frac{\alpha_s}{4\pi}\ (-9.20)\,\mathcal{O}_{ND}^{\rm lat}.
\end{aligned}
\tag{31}
$$

It is found that the coefficients of two new operators are $17.20/4\pi$ and $9.20/4\pi$, respectively, and are of $O(1)$. This means the possibility of large $O(a\alpha_s)$ correction for $\mathcal{O}_L^{\rm con}$ as in the case of axial vector current, though the lattice matrix elements of $\mathcal{O}_{LD}^{\rm lat}$ and $\mathcal{O}_{ND}^{\rm lat}$ are not yet known.

IV. DISCUSSION

In the previous section, we pointed out that the $\Delta B = 2$ operator might receive a large $O(a\alpha_s)$ correction. For the rigorous investigation of the $O(a\alpha_s)$ effect, we must rely on the future works. On the other hand, the previous simulations have not shown a clear cutoff dependence of the B_B and seem to imply that the vacuum saturate approximation (VSA) is plausible within 10% level around the used lattice cutoff scale ($\sim$ 2-3 GeV) [9,10,16,17]. In this section, therefore, we attempt assuming the VSA for the lattice matrix elements to estimate the $O(a\alpha_s)$ effects for the $B_Bf_B^2$ and B_B using the results of the previous sections and then investigate the consistency of our result with the previous simulations. Although this analysis is quite rough, we believe that it is possible to find some, at least, qualitative features.

Let us discuss the $O(a\alpha_s)$ correction for $\langle\overline{B^0}|\ \mathcal{O}_L^{\rm con}\ |B^0\rangle$ using the VSA. Under the VSA, the relevant lattice matrix elements take the following values,

$$
\begin{aligned}
&\langle\overline{B^0}|\ \mathcal{O}_L^{\rm lat}\ |B^0\rangle^{\rm (VSA)} = \langle\overline{B^0}|\ \mathcal{O}_R^{\rm lat}\ |B^0\rangle^{\rm (VSA)} \\
&= \langle\overline{B^0}|\ \mathcal{O}_N^{\rm lat}\ |B^0\rangle^{\rm (VSA)} = -\frac{8}{5}\langle\overline{B^0}|\ \mathcal{O}_S^{\rm lat}\ |B^0\rangle^{\rm (VSA)} = \frac{8}{3}\left(f_B^{(0)\rm lat}M_B\right)^2,
\end{aligned}
\tag{32}
$$

$$
\langle\overline{B^0}|\ \mathcal{O}_{LD}^{\rm lat}\ |B^0\rangle^{\rm (VSA)} = \langle\overline{B^0}|\ \mathcal{O}_{ND}^{\rm lat}|B^0\rangle^{\rm (VSA)} = -\delta f_B^{\rm lat}\frac{8}{3}\left(f_B^{(0)\rm lat}M_B\right)^2,
\tag{33}
$$

where $f_B^{(0)\rm lat}M_B \equiv \langle 0|J_{\gamma_5\gamma_4}^{(0)\rm lat}|\overline{B^0}\rangle$ and $\delta f_B^{\rm lat} \equiv \langle 0|J_{\gamma_5\gamma_4}^{(1)\rm lat}|\overline{B^0}\rangle/\langle 0|J_{\gamma_5\gamma_4}^{(0)\rm lat}|\overline{B^0}\rangle$. Substituting Eqs. (32) and (33) into Eq. (31), we obtain

$$
\begin{aligned}
&\langle\overline{B^0}|\ \mathcal{O}_L^{\rm con}\ |B^0\rangle \xrightarrow{\rm VSA} \langle\overline{B^0}|\ \mathcal{O}_L^{\rm con}\ |B^0\rangle^{\rm (VSA)} \\
&\quad = \frac{8}{3}\left(f_B^{(0)\rm lat}M_B\right)^2\left[\ 1+\frac{\alpha_s}{4\pi}\left(6\ln(a^2m_b^2)-2\ln(a^2\mu^2)-36.83+26.40\ \delta f_B^{\rm lat}\right)\ \right],
\end{aligned}
\tag{34}
$$

where the last term with $\delta f_B^{\rm lat}$ is essentially due to the $O(a\alpha_s)$ effect. We can use the data calculated by Ali Khan *et al.* in Ref. [15] to guess the value of $\delta f_B^{\rm lat}$ in the static limit. In our estimate, their finite mass results at $\beta = 6.0$ imply $\delta f_B^{\rm lat} \sim -0.5$ in the static limit. Using the coupling constants at the corresponding lattice with Lepage and Mackenzie prescription [13], $\alpha_s \sim 0.15$-0.25, we find that the magnitude of the $O(a\alpha_s)$ correction for $\langle\overline{B^0}|\ \mathcal{O}_L^{\rm con}\ |B^0\rangle^{\rm (VSA)}$ is very large, about 15-25%. Although this analysis is naive estimate of $O(a\alpha_s)$ correction

using the VSA, this suggests that there are large contribution from $O(a\alpha_s)$ correction for $\mathcal{O}_L^{\rm con}$ and the improvement of $O(a\alpha_s)$ should be necessarily included.

Now we turn to the B_B. The B_B is defined by

$$B_B = \frac{\langle \overline{B^0} | \ \mathcal{O}_L^{\rm con} \ | B^0 \rangle}{\frac{8}{3} \left(f_B M_B \right)^2}. \tag{35}$$

To improve the B_B in consistent way, we should include the $O(a\alpha_s)$ improvements of both of the numerator and denominator of Eq. (35). Substituting Eqs. (9) and (24) into Eq. (35) and linearizing the resulting expression in α_s according to the discussion of Ref. [16], we obtain the B_B as follows.

$$B_B = \sum_X \omega_X B_X^{\rm lat} - 2 \ \omega_1 \delta f_B^{\rm lat} B_L^{\rm lat},$$

where X runs over $\{L, S, N, R, LD, ND\}$,

$$\begin{aligned}
\omega_X &= \frac{Z_X}{(Z_{\gamma_5\gamma_4}^{(0)})^2}, \\
\omega_1 &= \frac{Z_{\gamma_5\gamma_4}^{(1)}}{Z_{\gamma_5\gamma_4}^{(0)}}, \\
B_X^{\rm lat} &= \frac{\langle \overline{B^0} | \ \mathcal{O}_X^{\rm lat} \ | B^0 \rangle}{\frac{8}{3} \left(f_B^{(0){\rm lat}} M_B \right)^2}.
\end{aligned}$$

In the VSA, using Eqs. (32) and (33) we obtain the following expression for B_B to $O(a\alpha_s)$.

$$\begin{aligned}
B_B \xrightarrow{\rm VSA} B_B^{\rm (VSA)} &= \left(\omega_L + \omega_R + \omega_N - \frac{5}{8} \omega_S \right) - \left(\omega_{LD} + \omega_{ND} + 2 \ \omega_1 \right) \delta f_B^{\rm lat}, \\
&= \left(1 + \frac{\alpha_s}{4\pi} D \right) - \left(\frac{\alpha_s}{4\pi} E \right) \delta f_B^{\rm lat},
\end{aligned} \tag{36}$$

where the term with $\delta f_B^{\rm lat}$ comes from the $O(a\alpha_s)$ improvements again. The coefficients D and E are given as follows.

$$D = \left[2 \ln \left(\frac{m_b^2}{\mu^2} \right) - \frac{14}{3} - \frac{2}{3} \left(d_1 + d_2 - d^I \right) - \frac{1}{3} c - \frac{1}{3} (v + v^I) + \frac{4}{3} \ (w + w^I) \right], \tag{37}$$

$$E = \frac{2}{3} \left[r \ (1 - c_{\rm sw}) \ln(a^2 \lambda^2) + U + U^I + V + V^I \right]. \tag{38}$$

In deriving Eqs. (37) and (38), there are some cancellations between the coefficients of the $\Delta B = 2$ operator and the axial vector current.

Now let us roughly estimate the $O(a\alpha_s)$ effect in the $B_B^{\rm (VSA)}$ numerically. When $r = c_{\rm sw} = 1$ is chosen, we obtain $D = 2 \ln(m_b^2/\mu^2) - 3.72$ and $E = -0.37$. Using the data of $\delta f_B^{\rm lat}$ and the coupling constants as before, we find that the $O(a\alpha_s)$ effect for the $B_B^{\rm (VSA)}$ is smaller than 1%. Of course such a drastic cancellation would not take place in reality due to deviations from VSA, but at least the present analysis suggests that there is a possibility of significant cancellation of $O(a\alpha_s)$ corrections in B_B. This is consistent with the observation from the previous simulations that there is not clear cutoff dependence in B_B.

V. CONCLUSION

In this paper, we reported the coefficients of the $O(a)$ operators which are newly induced at the $O(a\alpha_s)$ in the perturbative continuum-lattice operator matching of the heavy-light current and $\Delta B = 2$ operator. We also roughly estimated the $O(a\alpha_s)$ effect on the $B_B f_B^2$ and the B_B using the VSA in the lattice hadronic matrix elements. Although the $O(a\alpha_s)$ effect is significant in the determinations of f_B and $B_B f_B^2$, it seems that the effect is not so for B_B, at least, in this VSA analysis because the cancellation between the $O(a\alpha_s)$ effects in the numerator and denominator does work well. Therefore the previous works, which imply that there is no cutoff dependence in B_B, seem to be consistent with our analysis. Now, however, that the $O(a\alpha_s)$ improvement for the f_B has been already done, in order to calculate the B_B in a consistent way the $O(a\alpha_s)$ operators should be included in the calculation. For the precise determination, it is also required to include the finite mass correction into both calculations of the matrix element and the matching coefficients.

ACKNOWLEDGMENT

We would like to thank S. Hashimoto for useful discussion. T. O. and N. Y would also like to thank Summer Institute 98 at Kyoto supported by the Grant-in Aid for Scientific Research on Priority Areas (Physics of CP Violation) from the Ministry of Education, Science and Culture of Japan, where this work has started.

T. O. is supported by the Grants-in-Aid of the Ministry of Education (No.10740125).

APPENDIX A:

Here we show the wave function renormalization constants for each external quark line in each theory,

$$
\begin{aligned}
Z_q^{\rm con} &= 1 - \frac{\alpha_s}{4\pi} C_F \left[\mathcal{A} - \ln\left(\frac{\lambda^2}{\mu^2}\right) - \frac{1}{2} \right], \\
Z_b^{\rm con} &= 1 - \frac{\alpha_s}{4\pi} C_F \left[\mathcal{A} - \ln\left(\frac{m_b^2}{\mu^2}\right) - 2\ln\left(\frac{m_b^2}{\lambda^2}\right) + 4 \right], \\
Z_q^{\rm lat} &= 1 + \frac{\alpha_s}{4\pi} C_F \left[\ln(a^2\lambda^2) + f + f^I + u_0^{(2)} \right], \\
Z_b^{\rm lat} \equiv Z_Q^{\rm lat} = Z_\chi^{\rm lat} &= 1 + \frac{\alpha_s}{4\pi} C_F \left[-2\ln(a^2\lambda^2) + e^{(R)} \right],
\end{aligned}
$$

where f, f^I and $e^{(R)}$ can be found in Refs. [6,7]. In above equations $\mathcal{A} = 1/\epsilon + \ln(4\pi) - \gamma_E$ and $u_0^{(2)}$ is perturbative coefficient of the tadpole improvement factor defined by $u_0 = 1 + \alpha_s C_F u_0^{(2)}$. The coefficient $u_0^{(2)}$ is obtained through the calculation of the mean plaquette value or the critical hopping parameter, $u_0^{(2)} = -\pi^2$ or $u_0^{(2)} = -\left[4.4259 + 8.4327\ r - 4.8619\ c_{\rm sw}\right]$, respectively.

APPENDIX B:

Here we show the explicit forms of the integrands for U, U^I, V and V^I, which first appear in the $O(a\alpha_s)$. For shorthand notation, we define the following quantities,

$$
\begin{aligned}
\Delta_1 &= \sum_{\mu=1}^{4} \sin^2\left(\frac{l_\mu}{2}\right), \\
\Delta_2 &= \sum_{\mu=1}^{4} \sin^2(l_\mu) + 4r^2(\Delta_1)^2, \\
\Delta_1^{(3)} &= \sum_{\mu=1}^{3} \sin^2\left(\frac{l_\mu}{2}\right), \\
\Delta_2^{(3)} &= \sum_{\mu=1}^{3} \sin^2(l_\mu) + 4r^2(\Delta_1^{(3)})^2, \\
\Delta_4^{(3)} &= \sum_{\mu=1}^{3} \sin^2(l_\mu), \\
\Delta_5^{(3)} &= \sum_{\mu=1}^{3} \sin^2(l_\mu)\sin^2\left(\frac{l_\mu}{2}\right).
\end{aligned}
$$

Using the above convention,

$$
\begin{aligned}
U &= (4\pi)^2 \int_{-\pi}^{\pi} \frac{d^3 l}{(2\pi)^3} \left[\frac{1}{12\Delta_1^{(3)}\Delta_2^{(3)}} \left(3 + (3r^2-1)\Delta_1^{(3)}\right) \right. \\
&\quad \left. - \frac{1}{12\Delta_1^{(3)}(\Delta_2^{(3)})^2}(\Delta_4^{(3)} - 2\Delta_5^{(3)} + 2r^2\Delta_1^{(3)}\Delta_4^{(3)}) - \frac{2}{3(\vec{l}^2)^2}\theta(1-\vec{l}^2) \right] - \frac{16}{3}, \\
U^I &= (4\pi)^2 r^2 \int_{-\pi}^{\pi} \frac{d^3 l}{(2\pi)^3} \left[\frac{\Delta_4^{(3)}}{48\Delta_1^{(3)}\Delta_2^{(3)}} - \frac{1}{12(\Delta_2^{(3)})^2}(\Delta_4^{(3)} - 2\Delta_5^{(3)} + 2r^2\Delta_1^{(3)}\Delta_4^{(3)}) \right], \\
V &= (4\pi)^2 r \int_{-\pi}^{\pi} \frac{d^4 l}{(2\pi)^4} \left[-\frac{1}{4\Delta_2} \right. \\
&\quad - \frac{1}{12\Delta_1(\Delta_2)^2} \left\{ 12\left(1 + 2\Delta_1^{(3)} + 2(r^2-1)\Delta_1\right)(1 - \Delta_1 + \Delta_1^{(3)})\Delta_1^{(3)} \right. \\
&\quad \left.\left. + (\Delta_4^{(3)} - 2\Delta_5^{(3)} + 2r^2\Delta_4^{(3)}\Delta_1) \right\} + \frac{1}{(l^2)^2}\theta(1-l^2) \right], \\
V^I &= (4\pi)^2 r \int_{-\pi}^{\pi} \frac{d^4 l}{(2\pi)^4} \left[\frac{1}{12\Delta_1(\Delta_2)^2} \left\{ \left(1 + 2\Delta_1^{(3)} + 2(r^2-1)\Delta_1\right)\Delta_4^{(3)} \right. \right. \\
&\quad \left.\left. + (\Delta_4^{(3)} - 2\Delta_5^{(3)} + 2r^2\Delta_4^{(3)}\Delta_1) \right\} (1 - \Delta_1 + \Delta_1^{(3)}) - \frac{1}{l^2}\theta(1-l^2) \right].
\end{aligned}
$$

APPENDIX C:

Here we show the contribution from each diagram explicitly. In the continuum, the each contribution is as follows.

$$
\begin{aligned}
\mathcal{V}^{(a)}_{\rm con} &= \langle \mathcal{O}_L\rangle_0,\\
\mathcal{V}^{(b)}_{\rm con} &= \frac{\alpha_s}{4\pi}\left[\frac{10}{3}\mathcal{A} - \frac{10}{3}\ln\left(\frac{\lambda^2}{\mu^2}\right) - \frac{11}{3}\right]\langle \mathcal{O}_L\rangle_0 - \frac{\alpha_s}{4\pi}\ 8\ \langle \mathcal{O}_S\rangle_0 + \frac{\alpha_s}{4\pi}\ \frac{16\pi}{3a\lambda}\ \langle \mathcal{O}_{ND}\rangle_0,\\
\mathcal{V}^{(c)}_{\rm con} &= \frac{\alpha_s}{4\pi}\left[-\frac{4}{3}\mathcal{A} + \frac{4}{3}\ln\left(\frac{m_b^2}{\mu^2}\right) - \frac{2}{3}\ln\left(\frac{\lambda^2}{m_b^2}\right) - \frac{5}{3}\right]\langle \mathcal{O}_L\rangle_0,\\
\mathcal{V}^{(d)}_{\rm con} &= \frac{\alpha_s}{4\pi}\left[-\frac{4}{3}\mathcal{A} + \frac{4}{3}\ln\left(\frac{\lambda^2}{\mu^2}\right) - \frac{5}{3}\right]\langle \mathcal{O}_L\rangle_0,
\end{aligned}
$$

where (a) corresponds the tree diagram, (b) those with the gluon connecting the static and the light quarks, (c) those connecting the static quark and the static antiquark, and (d) those connecting the light quark and the light antiquark. And on the lattice,

$$
\begin{aligned}
\mathcal{V}^{(a)}_{\rm lat} &= \langle \mathcal{O}_L\rangle_0,\\
\mathcal{V}^{(b)}_{\rm lat} &= \frac{\alpha_s}{4\pi}\ \frac{10}{3}\ \left[-\ln(a^2\lambda^2) + d_1\right]\langle \mathcal{O}_L\rangle_0\\
&\quad + \frac{\alpha_s}{4\pi}\ 2\ \left[-d_2 + d^I\right]\langle \mathcal{O}_N\rangle_0\\
&\quad + \frac{\alpha_s}{4\pi}\ \frac{10}{3}\left[\ r(c_{\rm sw}-1)\ln(a^2\lambda^2) - (V+V^I)\right]\langle \mathcal{O}_{LD}\rangle_0\\
&\quad + \frac{\alpha_s}{4\pi}\ 2\left[\ \frac{8\pi}{3a\lambda} + (U+U^I)\right]\langle \mathcal{O}_{ND}\rangle_0,\\
\mathcal{V}^{(c)}_{\rm lat} &= \frac{\alpha_s}{4\pi}\ \frac{1}{3}\ \left[-\ 2\ln(a^2\lambda^2) + c\right]\langle \mathcal{O}_L\rangle_0,\\
\mathcal{V}^{(d)}_{\rm lat} &= \frac{\alpha_s}{4\pi}\ \frac{1}{3}\ \left[\ 4\ln(a^2\lambda^2) + (v+v^I)\right]\langle \mathcal{O}_L\rangle_0\\
&\quad + \frac{\alpha_s}{4\pi}\ \frac{4}{3}\ \left[-(w+w^I)\right]\langle \mathcal{O}_R\rangle_0.
\end{aligned}
\tag{C1}
$$

The calculation is straightforward though slightly lengthy. The full use of the equations of motion for the heavy and the light quarks and of the identities for γ matrices sometimes leads to simplification, in particular for the derivation of $\mathcal{V}^{(d)}_{\rm lat}$. We find that our result of $\mathcal{V}^{(d)}_{\rm lat}$ is inconsistent with Eqs. (B.16) and (B.25) of Ref. [7] provided the sign of the numerical values tabulated in Table 3 of the reference was correct, which has been already pointed out in Refs. [8,9].

REFERENCES

[1] S. Aoki *et al.* (JLQCD Collaboration), Phys. Rev. Lett. **80**, (1998) 5711; A. El-Khadra *et al.*, Phys. Rev. D**58**, (1998) 014506; C. Bernard *et al.* (MILC Collaboration), hep-ph/9806412; J. Hein, talk given at the XII International Symposium, Boulder, Colorado, to appear in Nucl. Phys. B, hep-lat/9809051.

[2] K-I. Ishikawa *et al.* (JLQCD Collaboration), talk given at the XII International Symposium, Boulder, Colorado, to appear in Nucl. Phys. B, hep-lat/9809152; K-I. Ishikawa *et al.* (JLQCD Collaboration), in preparation.

[3] For example, K-I. Ishikawa *et al.*, Phys. Rev. D**56**, (1997) 7028.

[4] T. Draper, plenary talk given at the XII International Symposium, Boulder, Colorado, to appear in Nucl. Phys. B, hep-lat/9810065.

[5] C. Morningstar and J. Shigemitsu, Phys. Rev. D**57**, (1998) 6741; C. Morningstar and J. Shigemitsu, hep-lat/9810047.

[6] J. Flynn, O. Hernandez and B. Hill, Phys. Rev. D**43**, (1991) 3709.

[7] A. Borrelli and C. Pittori, Nucl. Phys. B**385**, (1992) 502.

[8] M. Di Pierro, C. T. Sachrajda, Nucl. Phys. B**534**, (1998) 373.

[9] V. Gimenez and J. Reyes, hep-lat/9806023.

[10] A. K. Ewing *et al.* (UKQCD Collaboration), Phys. Rev. D**54**, (1996) 3526; V. Gimenez and G. Martinelli, Phys. Lett. B**398**, (1997) 135.

[11] B. Sheikholeslami and R. Wholert, Nucl. Phys. B**259**, (1985) 572.

[12] G. P. Lepage, Nucl. Phys. B(Proc. Suppl.) **26**, (1992) 45; A. X. El-Khadra, A. Kronfeld and P. Mackenzie, Phys. Rev. D**55**, (1997) 3933.

[13] G. P. Lepage and P. B. Mackenzie, Phys. Rev. D**48**, (1993) 2250.

[14] E. Eichten and B. Hill, Phys. Lett. B**234**, (1990) 511; *ibid*, B**240**, (1990) 193.

[15] A. Ali Khan *et al.*, Phys. Lett. B**427**, (1998) 132.

[16] J. Christensen, T. Draper and C. McNeile, Phys. Rev. D**56**, (1997) 6993.

[17] N. Yamada *et al.*, talk given at the XII International Symposium, Boulder, Colorado, to appear in Nucl. Phys. B, hep-lat/9809156.

TABLES

TABLE I. The numerical values of d^I, U, U^I, V and V^I for each value of r.

r	1.00	0.75	0.50	0.25	0
d^I	-4.14	-3.74	-3.12	-2.04	0
U	4.89	5.27	6.16	8.26	12.72
U^I	-0.29	-0.11	0.02	0.06	0
V	-7.14	-7.51	-7.72	-6.99	0
V^I	1.98	1.82	1.51	0.98	0

TABLE II. The results of the heavy-light current matching and H, H' and G for each Γ.

Γ	H	H'	G	$\zeta_\Gamma^{(0)}$	$\zeta_\Gamma^{(1)}$
1	4	1	1	$3\,\frac{3}{2}\ln(\mu^2/m_b^2) + \frac{3}{2}\ln(a^2m_b^2)$ - 2.25	0.56
γ_5	-4	-1	-1	$3\,\frac{3}{2}\ln(\mu^2/m_b^2) + \frac{3}{2}\ln(a^2m_b^2)$ - 8.41	9.76
γ_i	-2	-1	-1	$\frac{3}{2}\ln(a^2m_b^2)$ - 14.41	9.76
γ_4	-2	-1	1	$\frac{3}{2}\ln(a^2m_b^2)$ - 6.25	0.56
$\gamma_5\gamma_i$	2	1	1	$\frac{3}{2}\ln(a^2m_b^2)$ - 8.25	0.56
$\gamma_5\gamma_4$	2	1	-1	$\frac{3}{2}\ln(a^2m_b^2)$ - 12.41	9.76
σ_{4i}	0	1	-1	- $\frac{3}{2}\ln(\mu^2/m_b^2) + \frac{3}{2}\ln(a^2m_b^2)$ - 14.41	9.76
σ_{ij}	0	1	1	- $\frac{3}{2}\ln(\mu^2/m_b^2) + \frac{3}{2}\ln(a^2m_b^2)$ - 8.25	0.56